Indifference valuation and hedging of stock options in stochastic volatility models

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Abstract

We review the theoretical foundation for utility-based indifference-pricing of European claims on stocks driven by a stochastic volatility model [8] and explore the implementation of a non-linear finite difference scheme for solving the resultant pricing PDE. Specifically, we investigate the numerical solution for indifference-valuation of such claims under the Heston [6] stochastic volatility model.

1 Introduction

The pricing of European options under stochastic volatility is a central problem in modern finance. Various methodologies have been proposed over the years in the literature for the pricing of such options. The traditional view is that the markets select a particular pricing measure that is reflected in the prices of liquid traded assets, while more recent work broadly recasts the problem into that of either (i) portfolio optimisation or (ii) optimisation over pricing measures (see [9] for a comprehensive survey). In this review, we examine the portfolio optimisation approach of utility-based indifference-pricing of European claims under stochastic volatility.

Stochastic volatility models capture the volatility skew observed in many options markets and are popular extensions to the standard Black-Scholes pricing frameworks. While it successfully replicates option prices exhibiting implied volitility skews, such models also inadvertently introduce an additional source of uncertainty not traded in the market, resulting in incomplete markets.

In incomplete markets, there is no unique risk-neutral measure. Instead, there are infinitely many of them, each producing a valid no arbitrage price. The main challenge to pricing claims in incomplete markets is to choose an equivalent martingale measure (EMM) to produce the "right" price. The method to do so is to use arguments from mathematically precise and economically rational optimisation problems to derive the EMM and obtain the "right price" at which we should value our contingent claim. Any such method generally involves, a selection mechanism which determines why one valid EMM is preferred over the other.

The theory of utility-based indifference-pricing exploits the use of well-studied investor risk preferences encoded via the choice of suitable utility functions. The "right" EMM is deduced by optimizing a portfolio with respect to the utility function in a manner similar to the classic Merton's problem [7].

2 Preliminaries

In this section, we provide the basic theoretical setup and explain the mathematical rationale behind indifference-pricing.

2.1 Market Setup

We consider a dynamic market with a riskless bond B and a stock S. The stock price follows a diffusion process satisfying,

$$dS_t = \mu S_t dt + \sigma(Y_t, t) S_t dW_t$$

with the volatility term $\sigma(Y_t, t)$ modelled as driven by a correlated diffusion process Y satisfying,

$$dY_t = b(Y_t, t)dt + a(Y_t, t)(\rho dW_t + \sqrt{1-\rho^2} dW^{\perp})$$

The term $\rho \in (-1, 1)$ is the correlation coefficient and W and $W^{\perp}$ are independent Brownian motions on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by $((W_u, W_u^{\perp}); 0 \leq u \leq t)$. The interest rate on the riskless bond is assumed to be zero for ease of exposition and without loss of generality.

The derivative we are pricing is a European claim with payoff $g(S_T, Y_T)$ at time T. The usual assumptions are made on the functions $\sigma(\cdot, \cdot)$, $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ so as to guarantee unique solutions for S_t and Y_t, and we also require the payoff function $g(\cdot, \cdot)$ to be smooth and bounded.

Unfortunately, this requirement on $g(\cdot, \cdot)$ precludes the standard call and put payoffs. More careful handling of their discontinuities via regularization methods (such as that in [5]) is required before they would fit into this framework. Nevertheless, we investigate approximate payoffs which satisfy the smoothness conditions while at the same time, remain instructive in providing qualitative and quantitative insights on prices produced by indifference-pricing.

2.2 Rationale behind Indifference-Pricing

We now briefly discuss the classical portfolio optimisation problem formulated by Merton in [7]. Its significance is that indifference-pricing exploits two closely related Merton-type problems to deduce a rational choice for the price of our contigent claim.

2.2.1 Utility from Terminal Wealth

In Merton's utility from terminal wealth problem, an investor with initial endowment x dynamically rebalances his self-financing portfolio by trading in the stock and bond. The objective of the investor is to maximize his utility from terminal wealth.

Let $\pi_t = H_t S_t$ be the amount of wealth invested in the stock at time t, where H_t is the holdings in stock. The wealth dynamics for the investor at time t is thus given by

$$\begin{aligned} dX_t &= H_t dS_t + r(X_t - H_t S_t)dt \\ &= H_t S_t(\mu dt + \sigma(Y_t, t)dW_t) \qquad \text{(since r=0)} \\ &= \mu \pi_t dt + \sigma(Y_t, t)\pi_t dW_t \end{aligned}$$

We assume that the investor has risk preferences expressed by the exponential utility function,

$$U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}, \alpha \geq 0$$

The classical Merton problem, modified appropriately to account for stochastic volatility, is the optimisation problem with the value function,

$$V(x, y, t) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\alpha X_T)|X_t = x, Y_t = y\right]$$

where the investor seeks to maximize his utility of terminal wealth X_T. (Note : S appears only implicitly in π_t in the above formulation.)

2.2.2 A Thought Experiment

Suppose we now consider a slight modification to the problem where the investor needs to pay out one unit of the claim $g(S_T, Y_T)$ at time T. The new optimisation problem becomes,

$$\bar{V}(x, S, y, t) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\alpha(X_T - g(S_T, Y_T)))|X_t = x, S_t = S, Y_t = y\right]$$

For the same values of initial endowment x and volatility y at time t, we have $\bar{V}(x, S, y, t) \leq V(x, y, t)$, since there is an additional $g(S_T, Y_T)$ subtracted off from the terminal wealth.

Imagine the investor is given the option of choosing to optimize V versus $\bar{V}$ (for the same set of starting values x, S, y, t) so as to obtain the highest utility from either V or $\bar{V}$. Clearly, in the current setup, the investor would always choose to work on the problem corresponding to V.

How could we incentivise the investor to consider working on the problem for $\bar{V}$?

We could increase his initial endowment by some value $w \geq 0$, should he choose to work on $\bar{V}$. In other words, we modify the problem for $\bar{V}$ to read,

$$\bar{V}(x+w, S, y, t) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\alpha(X_T - g(S_T, Y_T)))|X_t = x, S_t = S, Y_t = y\right]$$

2.2.3 Deducing the Claim Price

Consider the value $\Delta V = V(x,y,t) - \bar{V}(x+w,S,y,t)$. As we increase the value of w from zero, ΔV decreases to a point where we eventually get equality for some $w = w^\star$. This makes it more attractive for the investor to consider working on the problem $\bar{V}$. At equality, where we have $V(x,y,t) = \bar{V}(x+w,S,y,t)$, it no longer matters which optimisation problem the investor chooses, since he is now able to extract the same level of optimal utility from either of them.

In short, the investor is now *indifferent* between the two optimisation problems. This is in essence, the motivating rationale behind the theory of utility-based indifference-pricing.

Since the investor has to be compensated with extra initial endowment $w^\star$ in order for him to take on the liability of an additional payout $g(S_T, Y_T)$ at time T, we must logically conclude that $w^\star$ is the value he would place on the claim g at time t. Hence we call $w^\star$ the *indifference price* of the claim g.

3 Indifference Prices

We formally introduce indifference-pricing and the governing equations from which we can compute options prices. For generality, we define the initial extra endowment as $h(x,S,y,t)$ to allow for general dependence on initial states.

The investor's value function for utility from terminal wealth with claim g payout at time T is given by,

$$u(x,S,y,t) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\alpha(X_T - g(S_T, Y_T)))|X_t = x, S_t = S, Y_t = y\right]$$

Note that in the above formulation, we view the investor as the *writer* of the claim g (since the $-g(S_T, Y_T)$ above suggests that the investor has to pay out the claim at T). An analogous setup is to adopt the point of view of a *buyer* of the claim, receiving $+g(S_T, Y_T)$ at T and paying (subtracting) $h_b(x,S,y,t)$ from initial endowment to buy the claim. The analysis from the point of view as buyer is similar and we would omit for brevity.

The **indifference price** $h(x,S,y,t)$ is defined as that for which the following holds,

$$V(x,y,t) = u(x + h(x,S,y,t), S, y, t) \tag{1}$$

Notice in the definition of h above that the argument x appears. Recall that x is the initial wealth of the investor, and we have just derived prices for our contingent claim which is dependent on initial wealth. This is undesirable because different investors, depending on their amount of initial wealth, would price the same claim differently! It runs in stark constrast to the wealth independent prices one would get from no-arbitrage pricing in complete markets.

Fortunately, the clever choice of exponential utlity leads to equations without any wealth dependence (as we shall see in (9)). For other choices of utility functions, wealth dependence continues to exist. In these cases, further extensions of the current theory is required to reconcile the issue.

3.1 Merton's problem under Stochastic Volatility

We consider the solution for $V(x, y, t)$. By Ito,

$$dV = \left(V_t + \mathcal{L}^{(x)}V + \mathcal{L}^{(y)}V + \rho\pi\sigma(y,t)a(y,t)V_{xy}\right)dt + \text{ local martingale}$$

where $\mathcal{L}^{(x)}$ and $\mathcal{L}^{(y)}$ are the infinitesimal generators of diffusion for X_t and Y_t respectively. By the martingale optimality principle, we know that V solves the Hamilton-Jacobi-Bellman (HJB) equation,

$$V_t + \mathcal{L}^{(y)}V + \sup_{\pi}\left[\mathcal{L}^{(x)}V + \rho\pi\sigma(y,t)a(y,t)V_{xy}\right] = 0 \tag{2}$$

We look for separable solutions of the form,

$$V(x, y, t) = -e^{-\lambda x}F(y, t) \tag{3}$$

The partial derivatives for V are,

$$\begin{aligned} V_t = -e^{-\lambda x}F_t, \quad V_x = \lambda e^{-\lambda x}F, \quad V_{xx} = -\lambda^2 e^{-\lambda x}F \\ V_{xy} = \lambda e^{-\lambda x}F_y, \quad V_y = -e^{-\lambda x}F_y, \quad V_{yy} = -e^{-\lambda x}F_{yy} \end{aligned}$$

Using first order conditions with respect to π for the supremum term in (2) gives,

$$\sigma^2(y,t)\pi V_{xx} + \rho\sigma(y,t)a(y,t)V_{xy} + \mu V_x = 0$$

and hence the optimal control $\pi^\star$ is given by,

$$\pi^\star = -\frac{\rho\sigma(y,t)a(y,t)V_{xy} + \mu u_x}{\sigma^2(y,t)V_{xx}}$$

Substituting $\pi^\star$ into (2) we get,

$$\begin{aligned} & V_t + \mathcal{L}^{(y)}V - \frac{1}{2}\frac{\rho^2 a^2(y,t)V_{xy}^2}{V_{xx}} - \frac{\rho\mu a(y,t)V_{xy}V_x}{\sigma(y,t)V_{xx}} - \frac{1}{2}\frac{\mu^2 V_x^2}{\sigma^2(y,t)V_{xx}} = 0 \\ \Rightarrow \qquad & F_t + \mathcal{L}^{(y)}F - \frac{\rho\mu a(y,t)}{\sigma(y,t)}F_y = \frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)}F + \frac{1}{2}\rho^2 a^2(y,t)\frac{F_y^2}{F} \end{aligned}$$

If we let $F = f^\delta$ with $\delta = \frac{1}{1-\rho^2}$, we get,

$$f_t + \mathcal{L}^{(y)}f - \frac{\rho\mu a(y,t)}{\sigma(y,t)}f_y = \frac{1}{2}\frac{\mu^2(1-\rho^2)}{\sigma^2(y,t)}f \tag{4}$$

The terminal condition for the above is $F(y,T) = f^\delta(y,T) = 1$, since we require that $V(x,y,T) = -e^{-\lambda x}$.

By Feynman-Kac's theorem, we know that f admits the following probabilistic representation,

$$f(y,t) = \mathbb{E}^{\mathbb{Q}^M}\left[\exp\left(-\int_t^T \frac{\mu^2(1-\rho^2)}{2\sigma^2(Y_s,s)}ds\right)\Bigg| Y_t = y\right] \tag{5}$$

The probability measure $\mathbb{Q}^M$ in the above is the minimal martingale measure [4] associated with the change of measure specified by the following Radon-Nicodym derivative,

$$\left.\frac{\mathrm{d}\mathbb{Q}^M}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\int_0^t \frac{\mu}{\sigma(Y_s,s)}dW_s\right)$$

where $\mathcal{E}(\cdot)$ is the Doléans exponential.

By Girsanov's theorem, the $\mathbb{Q}^M$ brownian motions associated with the above change of measure are,

$$W_t^{\mathbb{Q}^M} = W_t + \int_0^t \frac{\mu}{\sigma(Y_s,s)}dW_s, \qquad W_t^{\perp,\mathbb{Q}^M} = W_t^{\perp}$$

from which we get the new dynamics of S_t and Y_t under $\mathbb{Q}^M$,

$$\begin{aligned} dS_t &= \sigma(Y_t,t)dW_t^{\mathbb{Q}^M} \\ dY_t &= \left(b(Y_t,t) - \frac{\rho\mu a(Y_t,t)}{\sigma(Y_t,t)}\right)dt + a(Y_t,t)(\rho dW_t^{\mathbb{Q}^M} + \sqrt{1-\rho^2}dW_t^{\perp,\mathbb{Q}^M}) \end{aligned}$$

and these complete the requirements for the Feynman-Kac representation in equation (5).

Finally, we conclude that the solution to Merton's problem under stochastic volatility $V(x,y,t)$ is given by,

$$V(x,y,t) = -e^{-\lambda x}\left(\mathbb{E}^{\mathbb{Q}^M}\left[\exp\left(-\int_t^T \frac{\mu^2(1-\rho^2)}{2\sigma^2(Y_s,s)}ds\right)\Bigg| Y_t = y\right]\right)^{\frac{1}{1-\rho^2}}$$

3.2 Modified Merton's problem with Claim Payout

We proceed to derive the solution for the modified Merton's problem under stochastic volality with claim payout. The value function is u where u satisfies,

$$u(x,S,y,t) = \sup_{\pi_t\in\mathcal{A}} \mathbb{E}\left[-\exp(-\alpha(X_T - g(S_T,Y_T)))|X_t = x, S_t = S, Y_t = y\right]$$

Applying Ito's lemma, we get,

$$\begin{aligned} du = \Big(&u_t + \mathcal{L}^{(x)}u + \mathcal{L}^{(y)}u + \mathcal{L}^{(S)}u + \rho\pi_t a(Y_t,t)\sigma(Y_t,t)u_{xy} \\ &+\rho a(Y_t,t)\sigma(Y_t,t)Su_{Sy} + \pi_t\sigma^2(Y_t,t)Su_{Sx}\Big)\,dt \\ &+ \text{local martingale} \end{aligned}$$

where $\mathcal{L}^{(x)}$, $\mathcal{L}^{(y)}$ and $\mathcal{L}^{(S)}$ are the infinitesimal generators of diffusion for X_t, Y_t and S_t respectively. By the martingale optimality principle, we know that u solves the HJB equation,

$$\begin{aligned} u_t + \mathcal{L}^{(y)}u + \mathcal{L}^{(S)}u + \rho a(Y_t,t)\sigma(Y_t,t)Su_{Sy} \\ + \sup_{\pi}\left[\mathcal{L}^{(x)}u + \rho\pi_t a(Y_t,t)\sigma(Y_t,t)u_{xy} + \pi_t\sigma^2(Y_t,t)Su_{Sx}\right] = 0 \end{aligned} \tag{6}$$

We look for separable solutions of the form,

$$u(x,S,y,t) = -e^{-\lambda x}G(S,y,t) \tag{7}$$

The partial derivatives for u are,

$$\begin{gathered} u_t = -e^{-\lambda x}G_t, \quad u_x = \lambda e^{-\lambda x}G, \quad u_{xx} = -\lambda^2 e^{-\lambda x}G, \quad u_{xy} = \lambda e^{-\lambda x}G_y \\ u_y = -e^{-\lambda x}G_y, \quad u_{yy} = -e^{-\lambda x}G_{yy}, \quad u_{Sy} = -e^{-\lambda x}G_{Sy} \\ u_S = -e^{-\lambda x}G_S, \quad u_{SS} = -e^{-\lambda x}G_{SS}, \quad u_{Sx} = \lambda e^{-\lambda x}G_S \end{gathered}$$

Using first order conditions with respect to π for the supremum term in (6) gives,

$$\sigma^2(y,t)\pi u_{xx} + \rho\sigma(y,t)a(y,t)u_{xy} + \sigma^2(y,t)Su_{Sx} + \mu u_x = 0$$

and hence the optimal control $\pi^\star$ is given by,

$$\pi^\star = -\frac{\rho\sigma(y,t)a(y,t)u_{xy} + \sigma^2(y,t)Su_{Sx} + \mu u_x}{\sigma^2(y,t)u_{xx}}$$

Substituting $\pi^\star$ into (6) we get,

$$\begin{aligned} & u_t + \mathcal{L}^{(y)}u + \mathcal{L}^{(S)}u + \rho a(y,t)\sigma(y,t)Su_{Sy} \\ & -\frac{1}{2}\frac{\left(\rho\sigma(y,t)a(y,t)u_{xy} + \sigma^2(y,t)Su_{Sx} + \mu u_x\right)^2}{\sigma^2(y,t)u_{xx}} = 0 \\ \Rightarrow \quad & G_t + \mathcal{L}^{(y)}G + \mathcal{L}^{(S)}G + \rho a(Y_t,t)\sigma(Y_t,t)SG_{Sy} - \frac{\rho\mu a(y,t)}{\sigma(y,t)}G_y \\ & = \mu SG_S + \frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)}G + \frac{1}{2}\sigma^2S^2\frac{G_S^2}{G} + \rho\sigma(y,t)a(y,t)S\frac{G_SG_y}{G} + \frac{1}{2}\rho^2a^2(y,t)\frac{G_y^2}{G} \end{aligned}$$

Now let $G = e^{\phi}$. The partial derivatives of G in terms of ϕ are,

$$\begin{gathered} G_t = \phi_t G, \quad G_y = \phi_y G, \quad G_{yy} = \left(\phi_y^2 + \phi_{yy}\right)G \\ G_{Sy} = \left(\phi_{Sy} + \phi_s\phi_y\right)G, \quad G_S = \phi_S G, \quad G_{SS} = \left(\phi_S^2 + \phi_{SS}\right)G \end{gathered}$$

Substituting these into the PDE for G and continuing gives,

$$\begin{aligned} \phi_t + \mathcal{L}^{(y)}\phi + \frac{1}{2}(1-\rho^2)a^2(y,t)\phi_y^2 + \cancel{\mu S\phi_S} + \frac{1}{2}\sigma^2(y,t)S^2\phi_{SS} + \rho a(y,t)\sigma(y,t)S\phi_{Sy} \\ -\frac{\rho\mu a(y,t)}{\sigma(y,t)}\phi_y = \cancel{\mu S\phi_S} + \frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)} \end{aligned} \tag{8}$$

We note that there is a slight mistake in [8] for the above derivation involving the elimination of the term $\mu S\phi_S$, which was missed.

To conclude, we have the solution $u(x,S,y,t) = -e^{-\lambda x}e^{\phi(S,y,t)}$, where ϕ satisfies the PDE in (8).

3.3 The Indifference Price PDE

From the definition of the indifference price in (1) and the definitions of V and u in (3) and (7) respectively, we have

$$
\begin{aligned}
& V(x,y,t) = u(x + h(x,S,y,t), S, y, t) \\
\Rightarrow \quad & -e^{-\lambda x} f^{\delta}(y,t) = -e^{-\lambda(x+h(x,S,y,t))}e^{\phi(S,y,t)} \\
\Rightarrow \quad & f^{\delta}(y,t) = e^{-\lambda h(x,S,y,t)}e^{\phi(S,y,t)} \\
\Rightarrow \quad & h(x,S,y,t) = \frac{1}{\gamma}\ln\frac{e^{\phi(S,y,t)}}{f(y,t)^{1/(1-\rho^2)}}
\end{aligned}
\tag{9}
$$

Using the above expression for the indifference price h, we can express ϕ in terms of h and f to get,

$$
\phi(S,y,t) = \gamma h(x,S,y,t) + \frac{1}{1-\rho^2}\ln f(y,t)
$$

Notice in the above that h is independent of x (meaning that our wealth dependence vanishes) so we can write $h(S,y,t) \equiv h(x,S,y,t)$. The partial derivatives of ϕ are,

$$
\begin{aligned}
&\phi_t = \gamma h_t + \delta\frac{f_t}{f}, \quad && \phi_y = \gamma h_y + \delta\frac{f_y}{f}, \quad && \phi_{yy} = \gamma h_{yy} + \delta\left(\frac{f_{yy}}{f} - \left(\frac{f_y}{f}\right)^2\right) \\
&\phi_{Sy} = \gamma h_{Sy}, && \phi_S = \gamma h_S, && \phi_{SS} = \gamma h_{SS}
\end{aligned}
$$

Substituting the above into (8) gives,

$$
\begin{aligned}
&\gamma h_t + \delta \frac{f_t}{f} + b(y,t)\left(\gamma h_y + \delta \frac{f_y}{f}\right) + \frac{1}{2}a^2(y,t)\left(\gamma h_{yy} + \delta\left(\frac{f_{yy}}{f} - \left(\frac{f_y}{f}\right)^2\right)\right) \\
&+\frac{1}{2}(1-\rho^2)a^2(y,t)\left(\gamma h_y + \delta \frac{f_y}{f}\right)^2 + \frac{1}{2}\gamma\sigma^2(y,t)S^2 h_{SS} + \gamma\rho a(y,t)\sigma(y,t)S h_{Sy} \\
&\qquad\qquad -\frac{\rho\mu a(y,t)}{\sigma(y,t)}\left(\gamma h_y + \delta\frac{f_y}{f}\right) = \frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)} \\
&\Rightarrow \gamma\left[h_t + \mathcal{L}^{(y)} h + \frac{1}{2}(1-\rho^2)a^2(y,t)\left(\gamma h_y^2 + 2\delta \frac{h_y f_y}{f} + \frac{\delta^2}{\gamma}\left(\frac{f_y}{f}\right)^2\right)\right. \\
&\qquad\qquad \left. +\frac{1}{2}\sigma^2(y,t)S^2 h_{SS} + \rho a(y,t)\sigma(y,t)S h_{Sy} - \frac{\rho\mu a(y,t)}{\sigma(y,t)} h_y\right] \\
&\qquad\qquad +\frac{\delta}{f}\left[f_t + \mathcal{L}^{(y)} f - \frac{1}{2}a^2(y,t)\frac{f_y^2}{f} - \frac{\rho\mu a(y,t)}{\sigma(y,t)} f_y\right] = \frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)} \\
&\Rightarrow \gamma\left[h_t + \mathcal{L}^{(y)} h + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2 + a^2(y,t)\frac{h_y f_y}{f} + \frac{1}{2}\sigma^2(y,t)S^2 h_{SS}\right. \\
&\qquad \left. +\rho a(y,t)\sigma(y,t)S h_{Sy} - \frac{\rho\mu a(y,t)}{\sigma(y,t)} h_y\right] + \cancel{\frac{1}{2}a^2(y,t)\frac{\delta}{f}\frac{f_y^2}{f}} + \frac{\delta}{f}\left[\cancel{\frac{\rho\mu a(y,t)}{\sigma(y,t)}} f_y\right. \\
&\qquad\qquad \left. + \cancel{\frac{1}{2}\frac{\mu(1-\rho^2)}{\sigma^2(y,t)}} f - \cancel{\frac{1}{2}a^2(y,t)\frac{f_y^2}{f}} - \cancel{\frac{\rho\mu a(y,t)}{\sigma(y,t)}} f_y\right] = \cancel{\frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)}}
\end{aligned}
$$

where we substituted $f_t + \mathcal{L}^{(y)} f = \frac{\rho\mu a(y,t)}{\sigma(y,t)} f_y + \frac{1}{2}\frac{\mu^2(1-\rho^2)}{\sigma^2(y,t)} f$ from (4) to make the cancellations in the last line.

Finally we have the indifference price PDE,

$$
\begin{aligned}
h_t + \mathcal{L}^{(y)} h + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2 + a^2(y,t)\frac{h_y f_y}{f} + \frac{1}{2}\sigma^2(y,t)S^2 h_{SS} \\
+\rho a(y,t)\sigma(y,t)S h_{Sy} - \frac{\rho\mu a(y,t)}{\sigma(y,t)} h_y = 0
\end{aligned} \tag{10}
$$

with terminal condition $h(S,y,t) = g(S,y)$.

3.4 Summary

To recap what we have done so far,

1. We derived the PDE for the classic Merton's problem with stochastic volatility, writing the solution as $V(x,y,t) = -e^{-\lambda x} f(y,t)^{\frac{1}{1-\rho^2}}$, where f satisfies the PDE,

$$\begin{aligned} f_t + \mathcal{L}^{(y)} f - \frac{\rho\mu a(y,t)}{\sigma(y,t)} f_y &= \frac{1}{2}\frac{\mu^2(1-\rho^2)}{\sigma^2(y,t)} f \\ f(y,T) = 1 & \qquad \text{(terminal condition)} \end{aligned}$$

2. We derived the PDE for the modified Merton's problem with stochastic volatility and claim payout, writing the solution as $u(x,S,y,t) = -e^{-\lambda x} e^{\phi(S,y,t)}$, where ϕ satisfies the PDE,

$$\begin{aligned} \phi_t + \mathcal{L}^{(y)}\phi + \frac{1}{2}(1-\rho^2)a^2(y,t)\phi_y^2 + \frac{1}{2}\sigma^2(y,t)S^2\phi_{SS} + \rho a(y,t)\sigma(y,t)S\phi_{Sy} \\ -\frac{\rho\mu a(y,t)}{\sigma(y,t)}\phi_y = \frac{1}{2}\frac{\mu^2}{\sigma^2(y,t)} \\ \phi(S,y,T) = \gamma g(S,y) \qquad \text{(terminal condition)} \end{aligned}$$

3. Writing $V(x,y,t) = u(x+h(S,y,t),S,y,t)$ and using the PDEs above, we derived the PDE for the indifference price $h(S,y,t)$,

$$\begin{aligned} h_t + \mathcal{L}^{(y)}h + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2 + a^2(y,t)\frac{h_y f_y}{f} + \frac{1}{2}\sigma^2(y,t)S^2 h_{SS} \\ +\rho a(y,t)\sigma(y,t)Sh_{Sy} - \frac{\rho\mu a(y,t)}{\sigma(y,t)}h_y = 0 \\ h(S,y,T) = g(S,y) \qquad \text{(terminal condition)} \end{aligned}$$

4 Numerical Solutions for the Indifference Price

Generally, the indifference price PDE in (10) does not have explicit solutions. In this section, we explore numerical methods using finite differences for solving the PDE. To do so, we must first select a stochastic volatility model for the process Y_t.

4.1 The Heston Model

We use the well-known Heston model [6], which describes the volatility process Y_t with the following dynamics,

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}\left(\rho dW_t + \sqrt{1-\rho^2}dW_t^{\perp}\right)$$

The Heston model has the property that Y_t positive and mean-reverting to θ.

The corresponding stock and wealth processes are,

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{Y_t} S_t dW_t \\
dX_t &= \mu \pi_t dt + \pi_t \sqrt{Y_t} dW_t
\end{aligned}
$$

The PDE (4) for f then becomes,

$$
\begin{aligned}
f_t + (\kappa(\theta - y) - \rho\mu\xi)\, f_y + \frac{1}{2}\xi^2 y f_{yy} - \frac{1}{2}\frac{\mu^2(1-\rho^2)}{y} f &= 0 \\
f(y,T) &= 1
\end{aligned}
$$

and similarly the indifference price (10) becomes,

$$
\begin{aligned}
h_t + (\kappa(\theta - y) - \rho\mu\xi)\, h_y + \frac{1}{2}\xi^2 y h_{yy} + \frac{1}{2}\gamma(1-\rho^2)\xi^2 y h_y^2 + \xi^2 y \frac{h_y f_y}{f} \\
+ \frac{1}{2} y S^2 h_{SS} + \rho\xi y S h_{Sy} = 0 \\
h(S,y,T) = g(S,y)
\end{aligned}
$$

4.2 Discretizing the PDE for $f(y,t)$

For discretizing f, we use the implicit Euler method (unconditionally stable) with forward differences in the time dimension and central differences in the space dimension.

The approximate partial derivatives are,

$$
\begin{aligned}
f_t(y_n, t_m) &\approx \frac{f_n^{m+1} - f_n^m}{\Delta t} \\
f_y(y_n, t_m) &\approx \frac{f_{n+1}^m - f_{n-1}^m}{2\Delta y} \\
f_{yy}(y_n, t_m) &\approx \frac{f_{n+1}^m - 2f_n^m + f_{n-1}^m}{\Delta y^2}
\end{aligned}
$$

where $y_n = n \times \Delta y$, $t_m = m \times \Delta t$.

Substituting the above into the PDE for f gives,

$$
\begin{aligned}
\frac{f_n^{m+1} - f_n^m}{\Delta t} + (\kappa(\theta - n\Delta y) - \rho\mu\xi)\, \frac{f_{n+1}^m - f_{n-1}^m}{2\Delta y} + \frac{1}{2}\xi^2 n\Delta y \frac{f_{n+1}^m - 2f_n^m + f_{n-1}^m}{\Delta y^2} \\
- \frac{1}{2}\frac{\mu^2(1-\rho^2)}{n\Delta y} f_n^m = 0
\end{aligned}
$$

and after collecting terms and re-arranging we have f^{m+1} in terms of f^m,

$$\begin{aligned}
f_n^{m+1} = &-\frac{1}{2}\left[(\kappa(\theta - n\Delta y) - \rho\mu\xi) + n\xi^2\right]\frac{\Delta t}{\Delta y} f_{n+1}^m \\
&+ \left[1 + n\xi^2 + \frac{1}{2n}\mu^2(1-\rho^2)\right]\frac{\Delta t}{\Delta y} f_n^m \\
&+ \frac{1}{2}\left[(\kappa(\theta - n\Delta y) - \rho\mu\xi) - n\xi^2\right]\frac{\Delta t}{\Delta y} f_{n-1}^m \\
&= a_n^m f_{n+1}^m + b_n^m f_n^m + c_n^m f_{n-1}^m
\end{aligned}$$

where we abbreviate $f_n^m = f(n\Delta y, m\Delta t)$. The terminal condition is $f(n\Delta y, T) = 1$.

The plot in Figure 1 shows the graph of $f(t, y)$ for specific values of the parameters $\rho, \theta, \kappa, \xi$ and the evolution of the shape of f from $t = 1.0$ down to $t = 0.0$. We have projected the contours of f as it evolves through time onto the f-y plane for greater clarity.

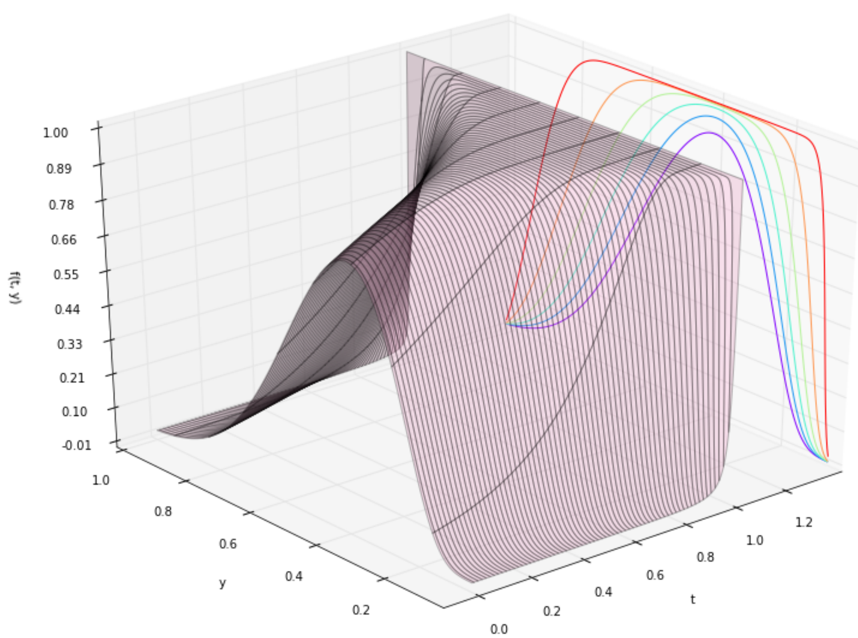

Figure 1: Plot of $f(t, y)$ for $\rho = 0.8$, $\theta = 0.2$, $\kappa = 2.0$, $\xi = 0.3$, $\mu = 0.2$.

4.2.1 Error Analysis

From taylor series expansion, we know that

$$\begin{aligned} f_n^{m+1} &= f_n^m + \Delta t \frac{\partial f}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 f}{\partial t^2} + \frac{1}{6}\Delta t^3 \frac{\partial^3 f}{\partial t^3} + \cdots \\ \Rightarrow \frac{f_n^{m+1} - f_n^m}{\Delta t} &= \frac{\partial f}{\partial t} + \frac{1}{2}\Delta t \frac{\partial^2 f}{\partial t^2} + O(\Delta t) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{f_{n+1}^m - f_{n-1}^m}{2\Delta y} &= \frac{\partial f}{\partial y} + \frac{1}{6}\Delta y^2 \frac{\partial^3 f}{\partial y^3} + O(\Delta y^2) \\ \frac{f_{n+1}^m - 2f_n^m + f_{n-1}^m}{\Delta y^2} &= \frac{\partial^2 f}{\partial y^2} + \frac{1}{12}\Delta y^2 \frac{\partial^4 f}{\partial y^4} + O(\Delta y^2) \end{aligned}$$

Using the above, and after subtracting the exact PDE from the finite difference approximation, we see that the finite difference scheme is consistent of order $O(\Delta t, \Delta y^2)$.

4.3 Discretizing the Indifference Price PDE

Similar to the method used for $f(t, y)$, we use forward differences in the time dimension and central differences in the space dimensions, the partial derivatives for h are,

$$\begin{aligned} h_t(S_i, y_j, t_m) &\approx \frac{h_{i,j}^{m+1} - h_{i,j}^m}{\Delta t}, \qquad h_S(S_i, y_j, t_m) \approx \frac{h_{i+1,j}^m - h_{i-1,j}^m}{2\Delta S} \\ h_y(S_i, y_j, t_m) &\approx \frac{h_{i,j+1}^m - h_{i,j-1}^m}{2\Delta y}, \quad h_{SS}(S_i, y_j, t_m) \approx \frac{h_{i+1,j}^m - 2h_{i,j}^m + h_{i-1,j}^m}{\Delta S^2} \\ h_{SS}(S_i, y_j, t_m) &\approx \frac{h_{i+1,j}^m - 2h_{i,j}^m + h_{i-1,j}^m}{\Delta S^2}, \quad h_{yy}(S_i, y_j, t_m) \approx \frac{h_{i,j+1}^m - 2h_{i,j}^m + h_{i,j-1}^m}{\Delta y^2} \\ h_{Sy}(S_i, y_j, t_m) &\approx \frac{(h_y)_{i+1,j} - (h_y)_{i-1,j}}{2\Delta S} \\ &= \frac{h_{i+1,j+1}^m - h_{i+1,j-1}^m - h_{i-1,j+1}^m + h_{i-1,j-1}^m}{4\Delta S \Delta y} \\ (h_y^2) &\approx \left(\frac{h_{i,j+1}^m - h_{i,j-1}^m}{2\Delta y} \right)^2 \\ &= \frac{\left(h_{i,j+1}^m\right)^2 - 2h_{i,j+1}^m h_{i,j-1}^m + \left(h_{i,j-1}^m\right)^2}{4\Delta y^2} \end{aligned}$$

Substituting the above into the PDE for h gives,

$$\frac{h_{i,j}^{m+1}-h_{i,j}^{m}}{\Delta t}+(\kappa(\theta-j\Delta y)-\rho\mu\xi)\frac{h_{i,j+1}^{m}-h_{i,j-1}^{m}}{2\Delta y}+\frac{1}{2}\xi^2 j\Delta y\frac{h_{i,j+1}^{m}-2h_{i,j}^{m}+h_{i,j-1}^{m}}{\Delta y^2}$$
$$+\frac{1}{2}\gamma(1-\rho^2)\xi^2 j\Delta y\frac{\left(h_{i,j+1}^{m}\right)^2-2h_{i,j+1}^{m}h_{i,j-1}^{m}+\left(h_{i,j-1}^{m}\right)^2}{4\Delta y^2}$$
$$+\xi^2 j\Delta y\frac{h_{i,j+1}^{m}-h_{i,j-1}^{m}}{2\Delta y}\frac{f_{j+1}^{m}-f_{j-1}^{m}}{2\Delta y f_j^m}+\frac{1}{2}j\Delta y(i\Delta S)^2\frac{h_{i+1,j}^{m}-2h_{i,j}^{m}+h_{i-1,j}^{m}}{\Delta S^2}$$
$$\rho\xi ij\Delta y\Delta S\frac{h_{i+1,j+1}^{m}-h_{i+1,j-1}^{m}-h_{i-1,j+1}^{m}+h_{i-1,j-1}^{m}}{4\Delta S\Delta y}=0$$

where we abbreviate $h_{i,j}^m=h(i\Delta S,j\Delta y,m\Delta t)$. The terminal condition is $h(i\Delta S,j\Delta y,T)=g(i\Delta S,j\Delta y)$. Rearranging terms we get,

$$\begin{aligned}
h_{i,j}^{m+1}&=h_{i,j}^{m}-\frac{1}{2}(\kappa(\theta-j\Delta y)-\rho\mu\xi)\frac{\Delta t}{\Delta y}\left[h_{i,j+1}^{m}-h_{i,j-1}^{m}\right]\\
&-\frac{1}{2}j\xi^2\frac{\Delta t}{\Delta y}\left[h_{i,j+1}^{m}-2h_{i,j}^{m}+h_{i,j-1}^{m}\right]\\
&-\frac{1}{8}\gamma j(1-\rho^2)\xi^2\frac{\Delta t}{\Delta y}\left[(h_{i,j+1}^{m})^2-2h_{i,j+1}^{m}h_{i,j-1}^{m}-(h_{i,j-1}^{m})^2\right]\\
&-\frac{1}{4}j\xi^2\frac{\Delta t}{\Delta y}\frac{\left[h_{i,j+1}^{m}-h_{i,j-1}^{m}\right]\left[f_{j+1}^{m}-f_{j-1}^{m}\right]}{f_j^m}\\
&-\frac{1}{2}ji^2\Delta y\Delta t\left[h_{i+1,j}^{m}-2h_{i,j}^{m}+h_{i-1,j}^{m}\right]\\
&-\frac{1}{4}ij\rho\xi\Delta t\left[h_{i+1,j+1}^{m}-h_{i+1,j-1}^{m}-h_{i-1,j+1}^{m}+h_{i-1,j-1}^{m}\right]\\
&=-\frac{1}{4}c_1^{i,j}h_{i+1,j+1}^{m}-\left[\frac{1}{2}\left(c_2^{i,j}+\left(1+\frac{1}{2}c_f^{i,j}\right)c_3^{i,j}\right)\right]\frac{\Delta t}{\Delta y}h_{i,j+1}^{m}\\
&+\frac{1}{4}c_1^{i,j}h_{i-1,j+1}^{m}-\frac{1}{2}c_4^{i,j}h_{i+1,j}^{m}+(1+c_3^{i,j}\frac{\Delta t}{\Delta y}+c_4^{i,j})h_{i,j}^{m}-\frac{1}{2}c_4^{i,j}h_{i-1,j}^{m}\\
&+\left[\frac{1}{2}\left(c_2^{i,j}-\left(1-\frac{1}{2}c_f^{i,j}\right)c_3^{i,j}\right)\right]\frac{\Delta t}{\Delta y}h_{i,j-1}^{m}-\frac{1}{4}c_1^{i,j}h_{i-1,j-1}^{m}\\
&-\frac{1}{8}c_5^{i,j}\frac{\Delta t}{\Delta y}\left[(h_{i,j+1}^{m})^2-2h_{i,j+1}^{m}h_{i,j-1}^{m}+(h_{i,j-1}^{m})^2\right]
\end{aligned}$$

where the constants $c_k^{i,j}$ are given by,

$$c_1^{i,j}=ij\rho\xi\Delta t,\quad c_2^{i,j}=(\kappa(\theta-j\Delta y)-\rho\mu\xi),\quad c_3^{i,j}=j\xi^2$$
$$c_4^{i,j}=ji^2\Delta y\Delta t,\quad c_5^{i,j}=j\gamma(1-\rho^2)\xi^2,\quad c_f^{i,j}=\frac{f_{j+1}^{m}-f_{j-1}^{m}}{f_j^m}$$

4.3.1 Error Analysis

The truncation errors for $h(S,y,t)$ for the first and second order derivatives in both time and space are similar to those for $f(t,y)$. In the following, we derive the new truncation error terms for the mixed derivative term h_{Sy}.

Using taylor series expansion in the S direction, we have,

$$h^m_{i\pm1,j+1} = h^m_{i,j+1} \pm \Delta S \frac{\partial h}{\partial S}\bigg|_{j+1} + \frac{1}{2}\ \Delta S^2 \frac{\partial^2 h}{\partial S^2}\bigg|_{j+1} \pm \frac{1}{6}\ \Delta S^3 \frac{\partial^3 h}{\partial S^3}\bigg|_{j+1} + \cdots$$

So we have,

$$h^m_{i+1,j\pm1} - h^m_{i-1,j\pm1} = 2\ \Delta S \frac{\partial h}{\partial S}\bigg|_{j\pm1} + \frac{1}{3}\ \Delta S^3 \frac{\partial^3 h}{\partial S^3}\bigg|_{j\pm1} + \cdots$$

Substitute into the expression for h_{Sy}, we get,

$$\begin{aligned}
h_{Sy} &\approx \frac{h^m_{i+1,j+1} - h^m_{i+1,j-1} - h^m_{i-1,j+1} + h^m_{i-1,j-1}}{4\Delta S\Delta y} \\
&= \frac{1}{4\Delta y}\left[2\ \frac{\partial h}{\partial S}\bigg|_{j+1} + \frac{1}{3}\ \Delta S^2 \frac{\partial^3 h}{\partial S^3}\bigg|_{j+1} - 2\ \frac{\partial h}{\partial S}\bigg|_{j-1} - \frac{1}{3}\ \Delta S^2 \frac{\partial^3 h}{\partial S^3}\bigg|_{j-1} + O(\Delta S^2)\right] \\
&= \frac{1}{4\Delta y}\left[2\left(\frac{\partial h}{\partial S}\bigg|_{j+1} - \frac{\partial h}{\partial S}\bigg|_{j-1}\right) + \frac{1}{3}\Delta S^2\left(\frac{\partial^3 h}{\partial S^3}\bigg|_{j+1} - \frac{\partial^3 h}{\partial S^3}\bigg|_{j-1}\right) + O(\Delta S^2)\right]
\end{aligned}$$

Now consider the terms $\frac{\partial h}{\partial S}\big|_{j\pm1}$ and $\frac{\partial^3 h}{\partial S^3}\big|_{j\pm1}$. Using taylor series expansion in the y direction, we have,

$$\begin{aligned}
\frac{\partial h}{\partial S}\bigg|_{j\pm1} &= \frac{\partial h}{\partial S}\bigg|_{j} \pm \Delta y\ \frac{\partial^2 h}{\partial S\partial y}\bigg|_{j} + \frac{1}{2}\Delta y^2\ \frac{\partial^3 h}{\partial S\partial y^2}\bigg|_{j} \pm \frac{1}{2}\Delta y^3\ \frac{\partial^4 h}{\partial S\partial y^3}\bigg|_{j} + \cdots \\
\frac{\partial^3 h}{\partial S^3}\bigg|_{j\pm1} &= \frac{\partial^3 h}{\partial S^3}\bigg|_{j} \pm \Delta y\ \frac{\partial^4 h}{\partial S^3\partial y}\bigg|_{j} + \frac{1}{2}\Delta y^2\ \frac{\partial^5 h}{\partial S^3\partial y^2}\bigg|_{j} \pm \frac{1}{2}\Delta y^3\ \frac{\partial^6 h}{\partial S^3\partial y^3}\bigg|_{j} + \cdots
\end{aligned}$$

So h_{Sy} becomes,

$$\begin{aligned}
h_{Sy} \approx \frac{1}{4\Delta y}&\Bigg[4\Delta y\frac{\partial^2 h}{\partial S\partial y} + 2\Delta y^3\frac{\partial^4 h}{\partial S\partial y^3} + O(\Delta y^3) \\
&+ \frac{1}{3}\Delta S^2\left(2\Delta y\frac{\partial^4 h}{\partial S^3\partial y} - \Delta y^3\frac{\partial^6 h}{\partial S^3\partial y^3} + O(\Delta y^3)\right) + O(\Delta S^2)\Bigg] \\
&= \frac{\partial^2 h}{\partial S\partial y} + O(\Delta S^2) + O(\Delta y^2)
\end{aligned}$$

Lastly, we note that the non-linear $(h_y)^2$ term contribute errors of order $O(\Delta y^2)$.

In conclusion, the scheme for $h(S, y, t)$ is consistent of order $O(\Delta t, \Delta S^2, \Delta y^2)$.

4.3.2 Solving the Non-linear Finite Difference for $h(S, y, t)$

Due to the presence of the non-linear $(h_y)^2$ term, instead of the usual linear system which we need to solve for the conventional black scholes PDE, we now have to solve a system of non-linear equations. This is required for every time-step in order to derive all the values of $h^m_{i,j}$ from the known values of $h^{m+1}_{i,j}$.

Specifically, we need to solve the non-linear system,

$$\begin{aligned} F_{i,j}(\mathbf{h}_{i,j}) = F_{i,j}(&h^{m+1}_{i,j}, h^m_{i+1,j+1}, h^m_{i,j+1}, h^m_{i-1,j+1}, h^m_{i+1,j}, h^m_{i,j}, \\ &h^m_{i-1,j}, h^m_{i+1,j-1}, h^m_{i,j-1}, h^m_{i-1,j-1}) = 0 \end{aligned}$$

for each grid point (i, j) in our finite difference grid. $F_{i,j}(\cdot)$ is the residual function defined by,

$$\begin{aligned} F_{i,j}(\mathbf{h}_{i,j}) = h^{m+1}_{i,j} & \\ &+ \frac{1}{4} c_1^{i,j} h^m_{i+1,j+1} \\ &+ \left[\frac{1}{2}\left(c_2^{i,j} - \left(1 + \frac{1}{2} c_f^{i,j}\right) c_3^{i,j}\right)\right] \frac{\Delta t}{\Delta y} h^m_{i,j+1} \\ &- \frac{1}{4} c_1^{i,j} h^m_{i-1,j+1} \\ &+ \frac{1}{2} c_4^{i,j} h^m_{i+1,j} \\ &- (1 + c_3^{i,j} \frac{\Delta t}{\Delta y} - c_4^{i,j}) h^m_{i,j} \\ &+ \frac{1}{2} c_4^{i,j} h^m_{i-1,j} \\ &- \left[\frac{1}{2}\left(c_2^{i,j} + \left(1 - \frac{1}{2} c_f^{i,j}\right) c_3^{i,j}\right)\right] \frac{\Delta t}{\Delta y} h^m_{i,j-1} \\ &+ \frac{1}{4} c_1^{i,j} h^m_{i-1,j-1} \\ &+ \frac{1}{8} c_5^{i,j} \frac{\Delta t}{\Delta y} \left[(h^m_{i,j+1})^2 - 2 h^m_{i,j+1} h^m_{i,j-1} + (h^m_{i,j-1})^2\right] \end{aligned}$$

4.4 Numerical Results

To solve the indifference price PDE, we researched techniques such as the Alternating-Direction Implicit method [3], which handles mixed derivative terms, but it is not easily extendible to handle non-linear terms. With no known effective techniques for solving non-linear finite differences, we explored the use of non-linear solvers in the Scipy package in Python.

We use the Newton-Krylov [1] method in Scipy for solving large-scale non-linear systems, along with the use of CUDA-based GPU acceleration available through the NumbaPro [2] package.

Our initial experiments reveal that it is infeasible to compute the residual function $F_{i,j}(\cdot)$ on the CPU for hundreds of thousands of grid points. Hence, we moved the computation of the residual function onto the GPU via NumbaPro's `@cuda.jit` api. With GPU support, we managed to complete the full finite difference computation within 35 minutes.

The following results were run from an iMac 3.4GHz Intel Core i7 with NVIDIA GeForce GTX 680MX using seamless Python-CUDA integration provided by NumbaPro. The finite difference S-y grid is 256×256 and 2^{16} steps for the time dimension. The full python source is available in Appendix C.

4.4.1 Approximate Digital Payoff

The plot in Figure 2 shows the graph of $h(0, S, y)$ for the same values of the parameters $\rho, \theta, \kappa, \xi, \mu, \gamma$ as those used for $f(t, y)$. We used the terminal value function $h(T, S, y) = g(S) = 1.0/(1 + e^{-\alpha(S-0.5)})$ where $\alpha = 500.0$ to approximate a digital call with strike 0.5 and payoff 1.0 when option is in the money.

The choice of this approximation is to satisfy the smoothness criteria required by the indifference price theory. Refer to Appendix A for the shape of $g(S)$, which corresponds to the graph for $h(t, S, y)$ for $t = 1.0$.

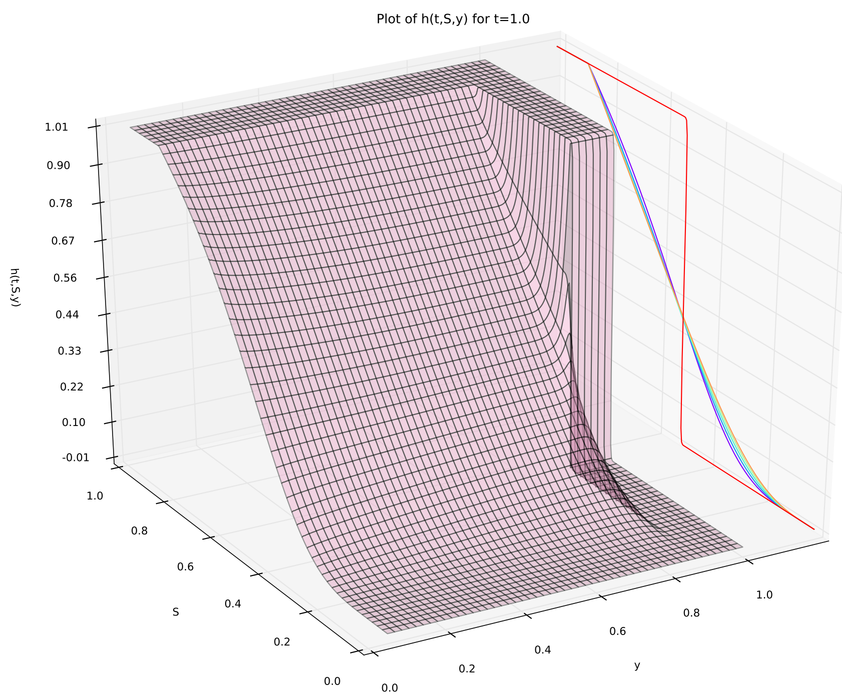

Figure 2: Plot of $h(0, S, y)$ for $\rho = 0.8$, $\theta = 0.2$, $\kappa = 2.0$, $\xi = 0.3$, $\mu = 0.2$, $\gamma = 0.1$.

In the plot above, we have projected the contours of h as y changes onto the h-y plane for greater clarity. From the figure, we can see the effects of the initial value $Y_0 = y$ on the shape of the price function at time 0.

See Appendix A for more graphs of $h(t, S, y)$ at various points in time for an intuitive feel of how the price function evolves as we step backwards through time from the terimal time T.

4.4.2 Approximate Put Payoff

We next use a smoothed put payoff with strike 0.5. The terminal value function used is,

$$h(T,S,y) = g(S) = 0.5 \times \left[\Phi\left(-\frac{logS/K - 0.5t}{\sqrt{t}}\right) - \frac{S}{K}\Phi\left(-\frac{logS/K + 0.5t}{\sqrt{t}}\right)\right]$$

for small values of t.

The plot in Figure 3 shows the graph of $h(0,S,y)$ for the same values of the parameters $\rho, \theta, \kappa, \xi, \mu, \gamma$ as those used for $f(t,y)$.

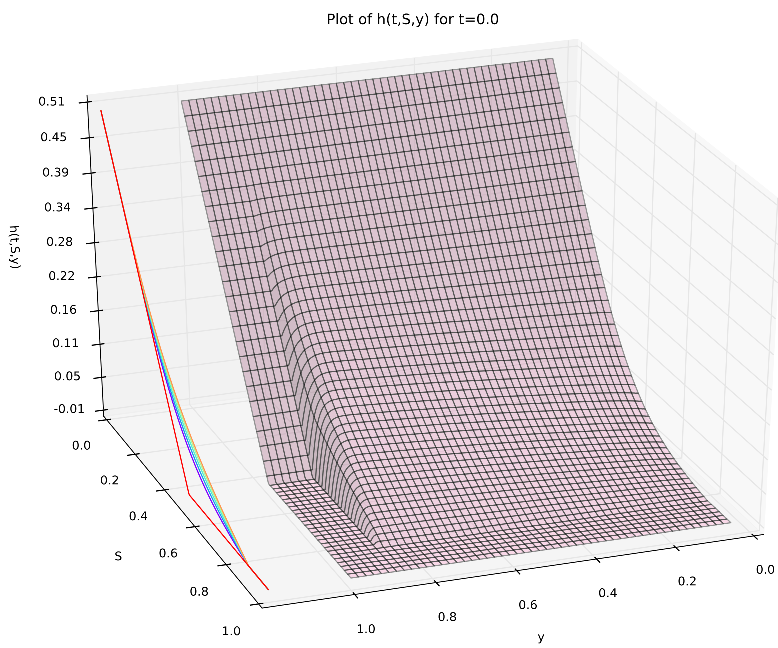

Figure 3: Plot of $h(0,S,y)$ for $\rho = 0.8$, $\theta = 0.2$, $\kappa = 2.0$, $\xi = 0.3$, $\mu = 0.2$, $\gamma = 0.1$.

Also included in Appendix B are more graphs of the evolution of $h(t,S,y)$ for the approximate put.

5 Conclusion and Future Work

We have provided a concise exposition on the theory of utility-based indifference-pricing and shown how to price a contingent claim for it under the Heston stochastic volatility model using non-linear finite differences.

The investigation provides a qualitative understanding of prices produced by indifference-pricing and also the computational complexities required by its numerical methods.

We note that the finite difference pricing logic is highly computation-intensive and impossible without the use of GPU methods. Thus, work should be done to investigate more advanced techniques for accelerating the pricing computation.

A Evolution of the Price Function for Approximate Digital Call

The plots below show the price function for the smoothed digital call at times $t = 1.0$, $t = 0.75$, $t = 0.5$ and $t = 0.25$. The terminal payoff function at $t = 1.0$ is $g(S) = 1.0/(1 + e^{-\alpha(S-0.5)})$ where $\alpha = 500.0$.

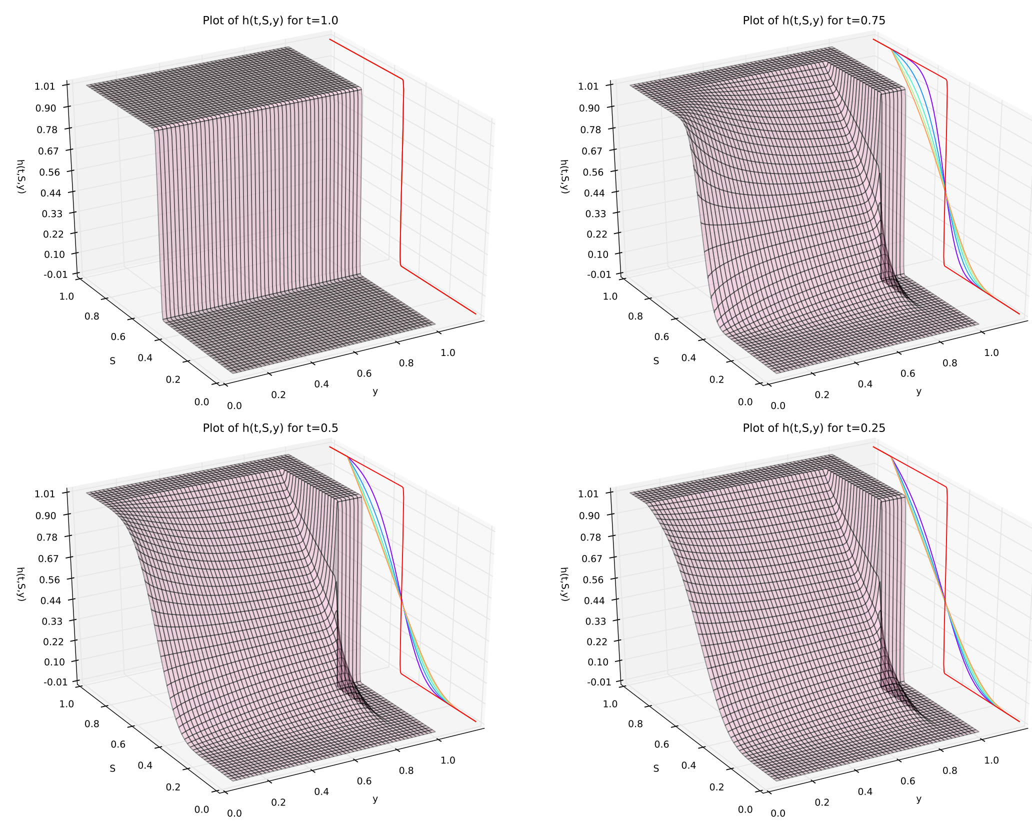

Figure 4: Time evolution of $h(t, S, y)$ for $\rho = 0.8$, $\theta = 0.2$, $\kappa = 2.0$, $\xi = 0.3$, $\mu = 0.2$, $\gamma = 0.1$.

B Evolution of the Price Function for Approximate Put

The plots below show the price function for the smoothed put at times $t = 1.0$, $t = 0.6$, $t = 0.3$ and $t = 0.0$. The terminal payoff function at $t = 1.0$ is $g(S) = 0.5 \times \left[\Phi\left(-\frac{logS/K-0.5t}{\sqrt{t}}\right) - \frac{S}{K}\Phi\left(-\frac{logS/K+0.5t}{\sqrt{t}}\right)\right]$ for small value of $t = 1 \times 10^{-6}$.

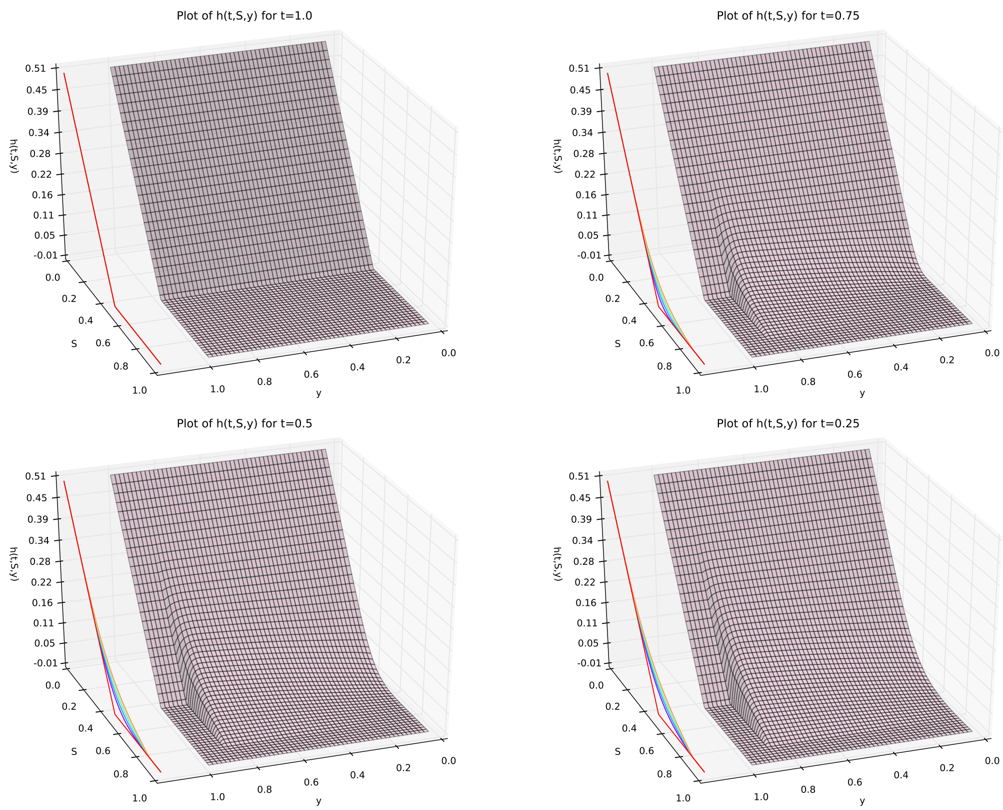

Figure 5: Time evolution of $h(t, S, y)$ for $\rho = 0.8$, $\theta = 0.2$, $\kappa = 2.0$, $\xi = 0.3$, $\mu = 0.2$, $\gamma = 0.1$.

C Python Finite Difference Code

```
from   functools import partial
from   numba import double, void
import numpy as np
from   scipy import sparse
from   scipy.sparse.linalg import spsolve
from   scipy.optimize import newton_krylov
from   numbapro import cuda
```

```
ONE_DIV_8 = 1 / 8.0

def payoff( S, alpha=500.0 ):
    return 1.0 / ( 1 + np.exp( -alpha * ( S - 0.5 ) ) )

#finite difference grid config
M = 65536 # this is for t axis
I = 255   # this is for S axis
J = 255   # this is for y axis

# heston params
rho   = 0.8
theta = 0.2
kappa = 2.0
xi    = 0.3
mu    = 0.2
gamma = 0.1

Ymin = 0.05
Ymax = 1.0
Smax = 10.0

Y = np.linspace( Ymin, 1.0, N+1 )
S = np.linspace( 0, Smax, I+1 )

# initialize the payoff function g(y)
grid_current = np.zeros( ( I+1, J+1 ) )
p = np.array( map( payoff, S ) )
for j in xrange( J+1 ):
    grid_current[:,j] = p

dt = T / float(M)      # timestep : dt
dy = Ymax / float( N ) # size of interval for y

dt_div_dy = dt/dy
dy_mul_dt = dt * dy

K1 = -0.5 * ( kappa * theta - rho * mu *xi ) * dt_div_dy
K2 = ( 0.5 - 0.5 * ( xi**2 )/dy ) * Y * dt_div_dy
A  = K1 + K2
A  = np.roll( A, -1 )

K3 = ( xi**2 ) * Y * ( dt_div_dy / dy )
K4 = ( 0.5 * ( mu**2 ) * ( 1 - rho**2 ) * dt ) / Y
B  = 1 + K3 + K4

K5 = -K1
K6 = ( -0.5 - 0.5 * ( xi**2 )/dy ) * Y * dt_div_dy
C  = K5 + K6
C  = np.roll( C, 1 )

V   = np.ones( N + 1 ) # initialize terminal value for f
Mat = sparse.spdiags([A, B, C], [-1,0,1], N + 1, N + 1 )

# precompute consts
xi_2      = xi**2
rho_2     = rho**2
rho_xi_dt = rho*xi*dt
rho_mu_xi = rho*mu*xi
gamma_1_rho2_xi2 = gamma * ( 1 - rho_2 ) * xi_2

@cuda.jit( double( double, double, double, double, double, double,
           double, double, double, double, double, double, double, ),
           device=True, inline=True )
def residual( i, j, cf, h_previous, h0, h1, h2, h3, h4, h5, h6, h7, h8 ):
    # h_previous is h_{i,j}^m
    # hnew is [i+1, j+1], [i,j+1], [i-1, j+1]   0, 1, 2
    #         [i+1, j]  , [i,j]  , [i-1,j]      3, 4, 5
    #         [i+1, j-1], [i,j-1], [i-1,j-1]    6, 7, 8
    c1 = i*j*rho_xi_dt
    c2 = ( kappa * ( theta - j*dy ) - rho_mu_xi )
    c3 = j * xi_2
    c4 = j * i * i * dy_mul_dt
    c5 = j * gamma_1_rho2_xi2

    t1  = -0.25 * c1 * h0
    t2  = -( 0.5 * ( c2 + ( 1 + 0.5 * cf ) * c3 ) ) * dt_div_dy * h1
    t3  =  0.25 * c1 * h2
    t4  = -0.5 * c4 * h3
    t5  =  ( 1 + c3 * dt_div_dy + c4 ) * h4
    t6  = -0.5 * c4 * h5
    t7  =  0.25 * c1 * h6
    t8  =  ( 0.5 * ( c2 - ( 1 - 0.5 * cf ) * c3 ) ) * dt_div_dy * h7
    t9  = -0.25 * c1 * h8
    t10 = -ONE_DIV_8 * c5 * dt_div_dy * ( ( h1 * h1 ) - (2 * h1 * h7) + ( h7 * h7 ) )

    return -h_previous + t1 + t2 +t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10

@cuda.jit( void( double[:,:], double[:,:], double[:], double[:,:] ) )
def _F( candidate, hgrid, fgrid, ret ):
```

```
    maxS, maxy = hgrid.shape
    i, j = cuda.grid(2)

    h0=h1=h2=h3=h4=h5=h6=h7=h8=candidate[i,j]
    if ( i+1 != maxS ) and ( j+1 != maxy ):
        h0 = candidate[i+1,j+1]
    if ( j+1 != maxy ):
        h1 = candidate[i,j+1]
    if ( i-1 >= 0 ) and ( j+1 != maxy ):
        h2 = candidate[i-1,j+1]
    if ( i+1 != maxS ):
        h3 = candidate[i+1,j]

    h4 = candidate[i,j]

    if ( i-1 >= 0 ):
        h5 = candidate[i-1,j]
    if ( i+1 != maxS ) and ( j-1 >= 0 ):
        h6 = candidate[i+1,j-1]
    if ( j-1 >= 0 ):
        h7 = candidate[i,j-1]
    if ( i-1 >= 0 ) and ( j-1 >= 0 ):
        h8 = candidate[i-1,j-1]

    fplus = fminus = fgrid[j]
    if ( j-1 >= 0 ):
        fminus = fgrid[j-1]

    if ( j+1 != maxy ):
        fplus = fgrid[j+1]

    cf = ( fplus - fminus ) / fgrid[j]

    ret[i,j] = residual( i, j, cf, hgrid[i,j], h0, h1, h2, h3, h4, h5, h6, h7, h8 )

# cuda params setup
tpb      = 32
blockdim = ( tpb, tpb )
griddim  = ( I / blockdim[0], J / blockdim[1] )
stream   = cuda.stream()

for i in xrange( M ):

    def FEntry( guess, hgrid, fgrid ):
        ret = np.zeros_like( hgrid )

        dV = cuda.to_device(V, stream) # data for f, only stream once
        dret = cuda.to_device(ret, stream) # the array for the results
        dhgrid = cuda.to_device( hgrid ) # the m+1 time-step grid

        dguess = cuda.to_device(guess, stream) # the guess answer
        _F[ griddim, blockdim, stream ]( dguess, dhgrid, dV, dret )
        dret.to_host( stream )
        stream.synchronize()
        return ret

    F   = partial( FEntry, hgrid=grid_current, fgrid=V )
    res = newton_krylov( F, grid_current, verbose=False )
    V   = spsolve( Mat, V )
    grid_current = res.copy()
```

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